

THE SPECTRAL DENSITY OF A PRODUCT OF SPECTRAL PROJECTIONS

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ABSTRACT. We consider the product of spectral projections

$$\Pi_\varepsilon(\lambda) = \mathbb{1}_{(-\infty, \lambda-\varepsilon)}(H_0) \mathbb{1}_{(\lambda+\varepsilon, \infty)}(H) \mathbb{1}_{(-\infty, \lambda-\varepsilon)}(H_0)$$

where H_0 and H are the free and the perturbed Schrödinger operators with a short range potential, $\lambda > 0$ is fixed and $\varepsilon \rightarrow 0$. We compute the leading term of the asymptotics of $\text{Tr } f(\Pi_\varepsilon(\lambda))$ as $\varepsilon \rightarrow 0$ for continuous functions f vanishing sufficiently fast near zero. Our construction elucidates calculations that appeared earlier in the theory of “Anderson’s orthogonality catastrophe” and emphasizes the role of Hankel operators in this phenomenon.

1. INTRODUCTION

1.1. Motivation from mathematical physics. This paper is partly motivated by a phenomenon called “Anderson’s orthogonality catastrophe”, which has been intensively discussed in the physics literature and has recently attracted attention from a mathematical perspective; see [GKM, KOS, GKMO] and the literature cited therein. Let H_0 and H be the free and the perturbed Schrödinger operators

$$H_0 = -\Delta, \quad H = -\Delta + V \quad \text{in } L^2(\mathbb{R}^d), \quad d \geq 1,$$

where, for the sake of simplicity, the real-valued potential V is assumed to be bounded and compactly supported. For $\lambda > 0$, consider the product of spectral projections

$$\Pi(\lambda) = \mathbb{1}_{(-\infty, \lambda)}(H_0) \mathbb{1}_{(\lambda, \infty)}(H) \mathbb{1}_{(-\infty, \lambda)}(H_0) \quad \text{in } L^2(\mathbb{R}^d). \quad (1.1)$$

One is interested in regularised versions of $\Pi(\lambda)$, obtained by replacing the step functions $\mathbb{1}_{(-\infty, \lambda)}$, $\mathbb{1}_{(\lambda, \infty)}$ by functions with disjoint supports. More precisely, we consider two types of regularisations of $\Pi(\lambda)$,

$$\Pi_\varepsilon^{(1)}(\lambda) = \mathbb{1}_{(-\infty, \lambda-\varepsilon)}(H_0) \mathbb{1}_{(\lambda+\varepsilon, \infty)}(H) \mathbb{1}_{(-\infty, \lambda-\varepsilon)}(H_0) \quad (1.2)$$

and

$$\Pi_\varepsilon^{(2)}(\lambda) = \psi_\varepsilon^-(H_0 - \lambda) \psi_\varepsilon^+(H - \lambda) \psi_\varepsilon^-(H_0 - \lambda), \quad (1.3)$$

where ψ_ε^\pm are continuous functions on \mathbb{R} that satisfy $0 \leq \psi_\varepsilon^\pm \leq 1$ and

$$\psi_\varepsilon^+(x) = \begin{cases} 0 & \text{if } x \leq \varepsilon, \\ 1 & \text{if } x \geq 2\varepsilon, \end{cases} \quad \psi_\varepsilon^-(x) = \begin{cases} 1 & \text{if } x \leq -2\varepsilon, \\ 0 & \text{if } x \geq -\varepsilon. \end{cases} \quad (1.4)$$

It is not difficult to see that the operators $\Pi_\varepsilon^{(j)}(\lambda)$, $j = 1, 2$, are trace class. On the other hand, $\Pi(\lambda)$ is typically not trace class and not even compact. We discuss the asymptotics of traces

$$\mathrm{Tr} f(\Pi_\varepsilon^{(j)}(\lambda)), \quad \varepsilon \rightarrow 0, \quad j = 1, 2, \quad (1.5)$$

where $f = f(t)$ is a continuous function which vanishes sufficiently fast as $t \rightarrow 0$.

It turns out that the asymptotics of the traces (1.5) is given in terms of the scattering matrix $S(\lambda)$ for the pair H_0, H at energy λ . Let $\{e^{i\theta_\ell(\lambda)}\}_{\ell=1}^L$ be the eigenvalues of $S(\lambda)$, enumerated with multiplicities taken into account. The scattering matrix is an operator in $L^2(\mathbb{S}^{d-1})$ for $d \geq 2$ and is a 2×2 matrix for $d = 1$; thus, $L = \infty$ for $d \geq 2$ and $L = 2$ for $d = 1$. Denote

$$a_\ell(\lambda) = \frac{1}{2} |e^{i\theta_\ell(\lambda)} - 1| = \left| \sin \frac{\theta_\ell(\lambda)}{2} \right| \in [0, 1], \quad \ell = 1, \dots, L. \quad (1.6)$$

For $j = 1, 2$ and for all continuous functions f , vanishing sufficiently fast at zero, we prove that

$$\lim_{\varepsilon \rightarrow 0} |\ln \varepsilon|^{-1} \mathrm{Tr} f(\Pi_\varepsilon^{(j)}(\lambda)) = \frac{1}{2\pi} \sum_{\ell=1}^L \int_{-\infty}^{\infty} f\left(\frac{a_\ell^2(\lambda)}{\cosh^2(\pi x)}\right) dx = \int_0^1 f(t) \mu_\lambda(t) dt, \quad (1.7)$$

where

$$\mu_\lambda(t) = \frac{1}{2\pi^2} \sum_{\ell=1}^L \frac{\mathbb{1}_{(0, a_\ell^2(\lambda))}(t)}{t \sqrt{1 - (t/a_\ell^2(\lambda))}}. \quad (1.8)$$

Of course, formula (1.8) is obtained from the first equality in (1.7) by means of a change of variable.

In [GKMO] a regularisation similar to $\Pi_\varepsilon^{(2)}(\lambda)$ is used; the authors essentially prove (1.7) for $f(t) = t^n$. (They state it with \geq instead of $=$, but in fact the proof contains the \leq case as well.) We believe that our construction is somewhat simpler than that of [GKMO]; we replace some of the heavy computations of [GKMO] by “soft” operator theoretic arguments. In particular, our proof highlights the key role of Hankel operators here. We will say more about it in Section 2.4.

In fact, our proof of the asymptotics (1.7) uses very little specific information about Schrödinger operators. For this reason, we state it as a general operator theoretic result for a pair of self-adjoint operators H_0, H , satisfying some standard assumptions of scattering theory.

1.2. Motivation from operator theory. In [Pu, PYa], the spectral structure of the operator $\Pi(\lambda)$ (see (1.1)) was studied in detail (for pairs of operators H_0, H satisfying some general assumptions of scattering theory). In particular, it was

proven that if $S(\lambda) \neq I$, then $\Pi(\lambda)$ has a non-trivial absolutely continuous spectrum, which consists of the union of intervals

$$\sigma_{\text{ac}}(\Pi(\lambda)) = \bigcup_{\ell=1}^L [0, a_\ell^2(\lambda)] ; \quad (1.9)$$

each interval contributes multiplicity one to the spectrum. Here $a_\ell(\lambda)$ are given by (1.6).

On the other hand, the regularisations $\Pi_\varepsilon^{(j)}(\lambda)$, $j = 1, 2$, of $\Pi(\lambda)$ are compact operators (see Lemma 2.3). Thus, it is reasonable to ask how the transition from the compact operators $\Pi_\varepsilon^{(j)}(\lambda)$ to the operator $\Pi(\lambda)$ with non-trivial absolutely continuous spectrum occurs and how the eigenvalues of $\Pi_\varepsilon^{(j)}(\lambda)$ concentrate to the spectral bands (1.9). Formulas (1.7), (1.8) partially answer this question: they give the eigenvalue density of $\Pi_\varepsilon^{(j)}(\lambda)$ as $\varepsilon \rightarrow 0$ as an explicit function $\mu_\lambda(y)$. Note that $\mu_\lambda(y)$ is given as a sum over ℓ , where each summand is supported on a single band $[0, a_\ell^2(\lambda)]$.

1.3. Notation. We denote by \mathbf{S}_p , $p \geq 1$, the standard Schatten class and by $\|\cdot\|_p$ the norm in this class. \mathbf{B} denotes the class of all bounded operators, \mathbf{S}_∞ is the class of all compact operators and $\|\cdot\|$ is the operator norm. If X and Y are two normal operators (possibly between different Hilbert spaces) such that

$$X|_{(\ker X)^\perp} \text{ is unitarily equivalent to } Y|_{(\ker Y)^\perp}$$

we write $X \approx Y$. It is well-known that $C^*C \approx CC^*$ for any bounded operator C . We will frequently use the fact that the relation $X \approx Y$ implies $\text{Tr } f(X) = \text{Tr } f(Y)$ for all continuous functions f with $f(0) = 0$. For a set $\Omega \subset \mathbb{R}$, we denote by $\mathbb{1}_\Omega$ the characteristic function of this set.

2. MAIN RESULT

2.1. Assumptions. Let H_0 and H be self-adjoint, lower semi-bounded operators in a Hilbert space \mathcal{H} such that

$$H = H_0 + V ,$$

where the perturbation V admits a factorization of the form

$$V = G^* V_0 G .$$

Here, \mathcal{K} is an auxiliary Hilbert space, V_0 is a bounded, self-adjoint operator in \mathcal{K} and G is a bounded operator from \mathcal{H} to \mathcal{K} satisfying

$$G(H_0 + M)^{-1/2} \in \mathbf{S}_\infty \quad (2.1)$$

for some constant $M > -\inf \text{spec } H_0$.

Remark 2.1. In fact, the boundedness of G is not necessary for our construction; we state it as a requirement here only in order to avoid inessential technical explanations.

We will need two assumptions: a global one (in spectral parameter) and a local one. The global assumption is

Assumption 2.2. We have

$$G(H_0 + M)^{-1/2} \in \mathbf{S}_\infty, \quad G(H_0 + M)^{-1/2-m} \in \mathbf{S}_{2p} \quad (2.2)$$

for some $p \geq 1$ and $M \geq 0$, $m \geq 0$.

(We have repeated the inclusion (2.1) here for the ease of further reference.) Let $\Pi_\varepsilon^{(1)}(\lambda)$, $\Pi_\varepsilon^{(2)}(\lambda)$ be as in (1.2), (1.3). Next, we denote

$$F_0(\lambda) := G\mathbb{1}_{(-\infty, \lambda)}(H_0)G^*, \quad F(\lambda) := V_0 G\mathbb{1}_{(-\infty, \lambda)}(H)G^*V_0.$$

In order to proceed, we need a simple intermediate result.

Lemma 2.3. *Suppose that Assumption 2.2 holds true. Then for all $\lambda \in \mathbb{R}$ and all $\varepsilon > 0$ we have*

$$F_0(\lambda) \in \mathbf{S}_p, \quad F(\lambda) \in \mathbf{S}_p, \quad \Pi_\varepsilon^{(1)}(\lambda) \in \mathbf{S}_p, \quad \Pi_\varepsilon^{(2)}(\lambda) \in \mathbf{S}_p.$$

The proof will be given in Section 3.

We fix some reference point $\lambda = \lambda_* \in \mathbb{R}$; our main result below concerns the spectral asymptotics of the operators $\Pi_\varepsilon^{(j)}(\lambda_*)$, $j = 1, 2$. Thus, our local assumption pertains to a neighbourhood of the point λ_* :

Assumption 2.4. There is a $\delta > 0$ such that the derivatives

$$F'_0(\lambda) = \frac{d}{d\lambda}F_0(\lambda), \quad F'(\lambda) = \frac{d}{d\lambda}F(\lambda)$$

exist in the \mathbf{S}_p norm for all λ in the interval $[\lambda_* - \delta, \lambda_* + \delta]$ and are Hölder continuous on this interval with some positive exponent $\varkappa > 0$.

By a version of Privalov's theorem, Assumption 2.4 implies that the operators

$$T_0(z) = G(H_0 - z)^{-1}G^*, \quad T(z) = V_0 G(H - z)^{-1}G^*V_0, \quad \text{Im } z > 0, \quad (2.3)$$

have limits $T_0(\lambda + i0)$, $T(\lambda + i0)$ in \mathbf{S}_p norm for λ in the open interval $(\lambda_* - \delta, \lambda_* + \delta)$, and these limits are Hölder continuous in λ on this interval. In other words, Assumption 2.4 implies a local version of the limiting absorption principle. Thus, by standard results of abstract scattering theory (see, e.g., [Ya1, Chapter 4]), the (local) wave operators for H_0 and H on the interval $(\lambda_* - \delta, \lambda_* + \delta)$ exist and the corresponding scattering matrix $S(\lambda)$ is well defined for λ in this interval.

Remark. In fact, we will only use the Hölder continuity of $F'_0(\lambda)$, $F'(\lambda)$ at the point $\lambda = \lambda_*$:

$$\begin{aligned} \|F'_0(\lambda) - F'_0(\lambda_*)\|_p &= O(|\lambda - \lambda_*|^\zeta) & \text{as } \lambda \rightarrow \lambda_*, \\ \|F'(\lambda) - F'(\lambda_*)\|_p &= O(|\lambda - \lambda_*|^\zeta) & \text{as } \lambda \rightarrow \lambda_*. \end{aligned}$$

The Hölder continuity as stated in Assumption 2.4 is needed only to ensure that the scattering matrix is well defined.

2.2. Main result. As in Section 1, we denote by $\{e^{i\theta_\ell(\lambda)}\}_{\ell=1}^L$, $L \leq \infty$, the eigenvalues of $S(\lambda)$, enumerated with multiplicities taken into account, and we use the notation $a_\ell(\lambda)$, see (1.6). Our main result is

Theorem 2.5. *Let Assumptions 2.2 and 2.4 hold true. Let $f(t) = t^p g(t)$ with g continuous on $[0, 1]$. Then for $j = 1, 2$, one has*

$$\lim_{\varepsilon \rightarrow +0} |\ln \varepsilon|^{-1} \operatorname{Tr} f(\Pi_\varepsilon^{(j)}(\lambda_*)) = \frac{1}{2\pi} \sum_{\ell=1}^L \int_{-\infty}^{\infty} f\left(\frac{a_\ell^2(\lambda_*)}{\cosh^2(\pi x)}\right) dx = \int_0^1 f(t) \mu_{\lambda_*}(t) dt, \quad (2.4)$$

where μ_{λ_*} is given by (1.8) with $\lambda = \lambda_*$.

Discussion.

- (1) As we shall see, Assumption 2.4 ensures that $\sum_{\ell=1}^L a_\ell^p(\lambda_*) < \infty$ (see Lemma 8.1), and so the series in (2.4) converges for the functions f as in the hypothesis of the theorem.
- (2) For $f(t) = t^n$, $n \geq p$, $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow +0} |\ln \varepsilon|^{-1} \operatorname{Tr} (\Pi_\varepsilon^{(j)}(\lambda_*))^n &= \sum_{\ell=1}^L a_\ell^{2n}(\lambda_*) \frac{1}{2\pi} \int_{-\infty}^{\infty} (\cosh^2(\pi x))^{-n} dx \\ &= \operatorname{Tr} \left(\frac{1}{2} |S(\lambda_*) - I| \right)^{2n} \frac{1}{2\pi} \int_{-\infty}^{\infty} (\cosh^2(\pi x))^{-n} dx. \end{aligned} \quad (2.5)$$

Using the change of variables $y = \cosh^2(\pi x)$, the integral in (2.5) can be explicitly computed,

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} (\cosh^2(\pi x))^{-n} dx &= \frac{1}{2\pi^2} \int_1^{\infty} y^{-n-\frac{1}{2}} (y-1)^{-\frac{1}{2}} dy \\ &= \frac{1}{2\pi^2} B(n, \tfrac{1}{2}) = \frac{n}{2\pi^2} \frac{((n-1)!)^2}{(2n)!} 2^{2n}, \end{aligned} \quad (2.6)$$

where $B(\cdot, \cdot)$ is the Beta function.

- (3) Of course, our conditions on f are far from optimal. For example, by a standard application of monotone convergence, the function f in (2.4) can be replaced by the characteristic function of any interval (α, β) , where $0 < \alpha < \beta \leq 1$. Then Theorem 2.5 can be interpreted as the convergence of the eigenvalue density of $\Pi_\varepsilon^{(j)}(\lambda_*)$ to the limiting density μ_{λ_*} given by the right side of (1.8).

- (4) Since our assumptions are symmetric in H and H_0 (in fact, (2.2) implies the same with H in place of H_0 , see Lemma 3.1), one can see that the statement, identical to Theorem 2.5, holds true for the spectral density of the operators

$$\mathbb{1}_{(-\infty, \lambda_* - \varepsilon)}(H) \mathbb{1}_{(\lambda_* + \varepsilon, \infty)}(H_0) \mathbb{1}_{(-\infty, \lambda_* - \varepsilon)}(H)$$

and

$$\psi_\varepsilon^-(H - \lambda_*) \psi_\varepsilon^+(H_0 - \lambda_*) \psi_\varepsilon^-(H - \lambda_*).$$

Since these operators are equivalent in the sense of the notion \approx introduced in Subsection 1.3 to the operators

$$\mathbb{1}_{(\lambda_* + \varepsilon, \infty)}(H_0) \mathbb{1}_{(-\infty, \lambda_* - \varepsilon)}(H) \mathbb{1}_{(\lambda_* + \varepsilon, \infty)}(H_0)$$

and

$$\tilde{\psi}_\varepsilon^+(H_0 - \lambda_*) \tilde{\psi}_\varepsilon^-(H - \lambda_*) \tilde{\psi}_\varepsilon^+(H_0 - \lambda_*),$$

(with $\tilde{\psi}_\varepsilon^+ = (\psi_\varepsilon^+)^{1/2}$ and $\tilde{\psi}_\varepsilon^- = (\psi_\varepsilon^-)^{1/2}$ which again are of the form (1.4)), we can see that the statement, identical to Theorem 2.5, holds true also for the spectral density of the latter operators.

- (5) If $m = 0$ in (2.2), then Theorem 2.5 holds true with $f(t) = t^{p/2}g(t)$ instead of $f(t) = t^p g(t)$. Indeed, in this case one can prove Lemma 3.2 with p instead of $2p$ and therefore Lemma 7.1 holds with $p/2$ instead of p .

Corollary 2.6. *Let Assumptions 2.2 and 2.4 hold true with $p = 1$. Then for $j = 1, 2$ we have the upper bound*

$$\limsup_{\varepsilon \rightarrow 0+} |\ln \varepsilon|^{-1} \ln \det (I - \Pi_\varepsilon^{(j)}(\lambda_*)) \leq \frac{1}{2\pi} \sum_{\ell=1}^L \int_{-\infty}^{\infty} \ln \left(1 - \frac{a_\ell^2(\lambda_*)}{\cosh^2(\pi x)} \right) dx. \quad (2.7)$$

The proof is given in Section 8.

Remark. (1) The integral on the right side of (2.7) can be computed and one obtains

$$\begin{aligned} \frac{1}{2\pi} \sum_{\ell=1}^L \int_{-\infty}^{\infty} \ln \left(1 - \frac{a_\ell(\lambda_*)^2}{\cosh^2(\pi x)} \right) dx &= -\frac{1}{\pi^2} \sum_{\ell=1}^L \arcsin^2 a_\ell(\lambda_*) \\ &= -\frac{1}{\pi^2} \operatorname{Tr} \arcsin^2 \frac{|S(\lambda_*) - I|}{2}. \end{aligned}$$

One way to see the first equality is to expand $\ln(1 - x) = -\sum_{n=1}^{\infty} x^n/n$ and to integrate term by term using (2.6). The claimed formula then follows from the expansion [GR, Eq. 1.645 2]

$$\sum_{n=1}^{\infty} 2^{2n-1} \frac{((n-1)!)^2}{(2n)!} a^{2n} = \arcsin^2 a, \quad |a| \leq 1.$$

This computation is very similar to a computation in [GKMO].

- (2) Under our assumptions, it is not possible to obtain any lower bound in (2.7).
 Indeed, a single eigenvalue $= 1$ of $\Pi_\varepsilon^{(j)}(\lambda_*)$ can make the determinant vanish.
 Such examples are easy to construct in the abstract setting discussed here.

2.3. Application to the Schrödinger operator. Let $H_0 = -\Delta$ in $L^2(\mathbb{R}^d)$, $d \geq 1$, and let the real-valued potential $V = V(x)$, $x \in \mathbb{R}^d$, satisfy

$$|V(x)| \leq C(1 + |x|)^{-\rho}, \quad \rho > 1. \quad (2.8)$$

Denote $H = H_0 + V$.

Lemma 2.7. *Assume (2.8). Then:*

- (i) *Assumption 2.2 is satisfied with any $p \geq 1$, $p > d/\rho$;*
- (ii) *Assumption 2.4 is satisfied with any $p > (d-1)/(\rho-1)$.*

Thus, the bound (2.8) ensures that Theorem 2.5 applies with any

$$p \geq 1, \quad p > \max\{d/\rho, (d-1)/(\rho-1)\}.$$

In particular, if $\rho > d$, then Corollary 2.6 applies.

Proof. (i) For $(1+2m)2p > d$ we shall verify the inclusion

$$|V|^{1/2}(-\Delta + I)^{-\frac{1}{2}-m} \in \mathbf{S}_{2p}.$$

Under our restrictions on p, ρ, m we have

$$\int_{\mathbb{R}^d} (|V(x)|^{1/2})^{2p} dx < \infty \quad \text{and} \quad \int_{\mathbb{R}^d} (|\xi|^2 + 1)^{-\frac{1}{2}-m} d\xi < \infty,$$

and therefore the above inclusion follows from the Kato–Seiler–Simon bound [S, Thm. 4.1].

(ii) By the standard (operator norm) limiting absorption principle, the derivatives $F'_0(\lambda)$, $F'(\lambda)$ exist for all $\lambda > 0$ and are given by

$$F'_0(\lambda) = \frac{1}{\pi} \operatorname{Im} T_0(\lambda + i0), \quad F'(\lambda) = \frac{1}{\pi} \operatorname{Im} T(\lambda + i0), \quad (2.9)$$

where T_0, T are defined in (2.3). The inclusion $F'_0(\lambda) \in \mathbf{S}_p$, $p > (d-1)/(\rho-1)$ and the Hölder continuity of this operator in \mathbf{S}_p norm are well known facts; see, e.g., [Ya2, Lemma 8.1.8]. The corresponding statements concerning $F'(\lambda)$ follow by an application of the resolvent identity. More specifically, one of the versions of the resolvent identity can be written as

$$T(z) = V_0 - (I + V_0 T_0(z))^{-1} V_0. \quad (2.10)$$

Taking the imaginary part here and using (2.10) again, we obtain

$$\begin{aligned} \operatorname{Im} T(z) &= (I + V_0 T_0(z))^{-1} V_0 (\operatorname{Im} T_0(z)) V_0 (I + T_0(z)^* V_0)^{-1} \\ &= (V_0 - T(z)) (\operatorname{Im} T_0(z)) (V_0 - T(z)^*). \end{aligned}$$

Passing to the limit $z \rightarrow \lambda + i0$ and using (2.9), we arrive at the identity

$$F'(\lambda) = (V_0 - T(\lambda + i0))F'_0(\lambda)(V_0 - T(\lambda + i0))^*.$$

This yields the required statements for $F'(\lambda)$. \square

2.4. Key ideas in the proof of Theorem 2.5. The main task is to prove Theorem 2.5 for the operator $\Pi_\varepsilon^{(1)}(\lambda_*)$; the statement for $\Pi_\varepsilon^{(2)}(\lambda_*)$ easily follows by some monotonicity arguments. In what follows, for simplicity of notation we take $\lambda_* = 0$ and set $\Pi_\varepsilon^{(j)} := \Pi_\varepsilon^{(j)}(0)$, $a_\ell := a_\ell(0)$. Our first step is a spectral localisation lemma (Lemma 7.1): we show that the operator $\Pi_\varepsilon^{(1)}$,

$$\Pi_\varepsilon^{(1)} = \mathbb{1}_{(-\infty, -\varepsilon)}(H_0) \mathbb{1}_{(\varepsilon, \infty)}(H) \mathbb{1}_{(-\infty, -\varepsilon)}(H_0), \quad (2.11)$$

can be replaced by the operator

$$\tilde{\Pi}_\varepsilon^{(1)} = \mathbb{1}_{(-\delta, -\varepsilon)}(H_0) \mathbb{1}_{(\varepsilon, \delta)}(H) \mathbb{1}_{(-\delta, -\varepsilon)}(H_0), \quad (2.12)$$

where δ is defined in Assumption 2.4. This is a standard argument using resolvent identities and some functional calculus for self-adjoint operators. Next, the key step is the product representation (Lemma 6.2)

$$\mathbb{1}_{(\varepsilon, \delta)}(H) \mathbb{1}_{(-\delta, -\varepsilon)}(H_0) = \mathcal{Z}_\varepsilon (\mathcal{Z}_\varepsilon^{(0)})^*. \quad (2.13)$$

Here, the operators $\mathcal{Z}_\varepsilon, \mathcal{Z}_\varepsilon^{(0)} : L^2(\mathbb{R}_+, \mathcal{K}) \rightarrow \mathcal{H}$ are defined by

$$\mathcal{Z}_\varepsilon^{(0)} f = \int_0^\infty e^{tH_0} \mathbb{1}_{(-\delta, -\varepsilon)}(H_0) G^* f(t) dt, \quad (2.14)$$

$$\mathcal{Z}_\varepsilon f = \int_0^\infty e^{-tH} \mathbb{1}_{(\varepsilon, \delta)}(H) G^* V_0 f(t) dt. \quad (2.15)$$

Of course, from (2.13) it follows that the operator $\tilde{\Pi}_\varepsilon^{(1)}$ admits the factorization

$$\tilde{\Pi}_\varepsilon^{(1)} = \mathcal{Z}_\varepsilon^{(0)} (\mathcal{Z}_\varepsilon)^* \mathcal{Z}_\varepsilon (\mathcal{Z}_\varepsilon^{(0)})^*.$$

Further, it turns out that the products $K_\varepsilon = (\mathcal{Z}_\varepsilon)^* \mathcal{Z}_\varepsilon$ and $K_\varepsilon^{(0)} = (\mathcal{Z}_\varepsilon^{(0)})^* \mathcal{Z}_\varepsilon^{(0)}$ are integral Hankel type operators in $L^2(\mathbb{R}_+, \mathcal{K})$. That is, these operators have the form

$$K_\varepsilon : f \mapsto \int_0^\infty k_\varepsilon(t+s) f(s) ds, \quad t > 0, \quad (2.16)$$

where $k_\varepsilon = k_\varepsilon(t)$ is some operator valued function, called the *kernel* of K_ε . As $\varepsilon \rightarrow 0$, the operators $K_\varepsilon, K_\varepsilon^{(0)}$ can be approximated in \mathbf{S}_p norm by some “model Hankel operators” with explicit integral kernels. By using the \approx relation (see Section 1.3), this allows us to reduce the problem to computing traces of powers of these model Hankel operators (Lemma 4.1). The latter turns out to be a relatively easy task.

Remark 2.8. In [GKMO], the authors consider the traces $\text{Tr}(\Pi_\varepsilon(\lambda))^n$ for the regularisation $\Pi_\varepsilon(\lambda)$ similar to our $\Pi_\varepsilon^{(2)}(\lambda)$. Through a series of transformations, the computation of the leading term of the asymptotics of this trace is reduced to the evaluation of some explicit multiple (n -fold) integral. It is curious that Hankel operators do appear in [GKMO], but only in passing, as a tool for evaluation of this integral. One of the points of this work is to emphasize that Hankel operators are at the heart of the matter here.

Remark 2.9. Much of the technique of the paper is borrowed from [Pu, PYa]. Crucially, the idea of the factorization (2.13) and the analysis of the operators K_ε and $K_\varepsilon^{(0)}$ comes from [Pu].

2.5. The structure of the paper. In Section 3 we prove the preliminary Lemma 2.3. In Sections 4 and 5 we prepare some auxiliary statements concerning Hankel operators. More precisely, in Section 4 we compute the asymptotics of traces of powers of a model Hankel operator and in Section 5 we present some \mathbf{S}_p class estimates for operator valued integral Hankel operators. In Section 6 we analyse the operators $\mathcal{Z}_\varepsilon, \mathcal{Z}_\varepsilon^{(0)}$, see (2.14), (2.15). In Section 7 we prove the spectral localization lemma, which reduces the analysis of $\Pi_\varepsilon^{(1)}$ to that of $\tilde{\Pi}_\varepsilon^{(1)}$, see (2.11), (2.12). In Section 8 we prove the main results of the paper.

3. PROOF OF LEMMA 2.3

Here we prepare some auxiliary statements which will be required in the proof of the spectral localization lemma (= Lemma 7.1) and prove Lemma 2.3.

Lemma 3.1. *Let Assumption 2.2 hold true. Then*

$$G(H + M)^{-\frac{1}{2}-m} \in \mathbf{S}_{2p}$$

for all $M > -\inf \sigma(H)$ and the same exponents m, p as in (2.2).

Proof. This is a straightforward adaptation of the argument of the proof of [RS, Theorem XI.12], where a variant of the above statement was proven for $p = 1/2$. For completeness, below we outline the proof. Choose $M > -\min\{\inf \sigma(H_0), \inf \sigma(H)\}$ sufficiently large so that

$$\|(H_0 + M)^{-\frac{1}{2}}V(H_0 + M)^{-\frac{1}{2}}\| \leq r < 1. \quad (3.1)$$

In order to make our formulas below more readable, set $h_0 = H_0 + M$ and $h = H + M$. Write

$$Gh^{-m-\frac{1}{2}} = Gh^{-m}h_0^{-\frac{1}{2}}h_0^{\frac{1}{2}}h^{-\frac{1}{2}}.$$

Since the operators h_0 and h have the same form domain, the product $h_0^{\frac{1}{2}}h^{-\frac{1}{2}}$ is bounded. We see that it suffices to prove the inclusion

$$Gh^{-m}h_0^{-\frac{1}{2}} \in \mathbf{S}_{2p}. \quad (3.2)$$

We have

$$h^{-1} = h_0^{-\frac{1}{2}} \left\{ \sum_{j=0}^{\infty} (h_0^{-\frac{1}{2}}(-V)h_0^{-\frac{1}{2}})^j \right\} h_0^{-\frac{1}{2}},$$

where by (3.1) the series converges in the operator norm. It follows that

$$Gh^{-m}h_0^{-\frac{1}{2}} = Gh_0^{-\frac{1}{2}} \sum_k \prod_{i=1}^k (h_0^{-\frac{1}{2}}(-V)h_0^{-\frac{1}{2}-\ell_i}), \quad (3.3)$$

where the sum is taken over the set of terms with $\ell_1 + \dots + \ell_k = m$. By interpolation between the two inclusions in (2.2) we obtain

$$Gh_0^{-\frac{1}{2}-\ell} \in \mathbf{S}_{2pm/\ell}, \quad 0 < \ell \leq m,$$

and therefore, using the Hölder inequality for Schatten classes, we see that each term in (3.3) satisfies

$$\prod_{i=1}^k (h_0^{-\frac{1}{2}}(-V)h_0^{-\frac{1}{2}-\ell_i}) \in \mathbf{S}_{2p}, \quad \ell_1 + \dots + \ell_k = m.$$

Moreover, as in [RS, Theorem XI.12], using condition (3.1), we obtain the estimate

$$\left\| \prod_{i=1}^k (h_0^{-\frac{1}{2}}(-V)h_0^{-\frac{1}{2}-\ell_i}) \right\|_{2p} \leq Cr^k$$

for each term in the series (3.3) over k . It follows that the series in (3.3) converges absolutely in the norm of \mathbf{S}_{2p} . Thus, we obtain (3.2). \square

Lemma 3.2. *Let Assumption 2.2 hold true. Then for any $\lambda_1 < \lambda_2$, we have*

$$\mathbb{1}_{(-\infty, \lambda_1)}(H_0)\mathbb{1}_{(\lambda_2, \infty)}(H) \in \mathbf{S}_{2p}, \quad (3.4)$$

$$\mathbb{1}_{(-\infty, \lambda_1)}(H)\mathbb{1}_{(\lambda_2, \infty)}(H_0) \in \mathbf{S}_{2p}. \quad (3.5)$$

Proof. First note that by Assumption 2.2 and by Lemma 3.1, we have

$$G\mathbb{1}_{(-\infty, \lambda)}(H_0) \in \mathbf{S}_{2p}, \quad G\mathbb{1}_{(-\infty, \lambda)}(H) \in \mathbf{S}_{2p}, \quad \forall \lambda \in \mathbb{R}. \quad (3.6)$$

Next, let $\lambda_{\min} < \min\{\inf \sigma(H_0), \inf \sigma(H)\}$. Choose a function $\psi \in C_0^\infty(\mathbb{R})$ such that

$$\psi(x) = \begin{cases} 0 & \text{for } x \leq \lambda_{\min} - 1 \text{ and for } x \geq \lambda_2, \\ 1 & \text{for } \lambda_{\min} \leq x \leq \lambda_1. \end{cases}$$

Then

$$\mathbb{1}_{(-\infty, \lambda_1)}(H_0)\psi(H_0) = \mathbb{1}_{(-\infty, \lambda_1)}(H_0) \quad \text{and} \quad \psi(H)\mathbb{1}_{(\lambda_2, \infty)}(H) = 0,$$

and therefore

$$\mathbb{1}_{(-\infty, \lambda_1)}(H_0)\mathbb{1}_{(\lambda_2, \infty)}(H) = \mathbb{1}_{(-\infty, \lambda_1)}(H_0)(\psi(H_0) - \psi(H))\mathbb{1}_{(\lambda_2, \infty)}(H). \quad (3.7)$$

Since $\psi \in C_0^\infty(\mathbb{R})$, we can use a standard method based on almost analytic continuation of ψ (see e.g. [DS, Chapter 8]) to represent it as

$$\psi(\lambda) = \int_{\mathbb{C}} \frac{\nu(z)}{\lambda - z} dL(z),$$

where $dL(z)$ is the 2-dimensional Lebesgue measure in \mathbb{C} , and ν is some function, compactly supported in \mathbb{C} and satisfying the estimate

$$\nu(z) = O(|\operatorname{Im} z|^N), \quad \operatorname{Im} z \rightarrow 0, \quad \forall N > 0. \quad (3.8)$$

Then, by the resolvent identity,

$$\mathbb{1}_{(-\infty, \lambda_1)}(H_0)(\psi(H_0) - \psi(H)) = \int_{\mathbb{C}} \mathbb{1}_{(-\infty, \lambda_1)}(H_0)(H_0 - z)^{-1} V(H - z)^{-1} \nu(z) dL(z). \quad (3.9)$$

Let us prove that the above integral converges absolutely in \mathbf{S}_{2p} . We have

$$\begin{aligned} & \mathbb{1}_{(-\infty, \lambda_1)}(H_0)(H_0 - z)^{-1} V(H - z)^{-1} \\ &= (H_0 - z)^{-1} (G \mathbb{1}_{(-\infty, \lambda_1)}(H_0))^* V_0 (G(H - i)^{-1}) (H - i)(H - z)^{-1}, \end{aligned}$$

and therefore

$$\begin{aligned} & \|\mathbb{1}_{(-\infty, \lambda_1)}(H_0)(H_0 - z)^{-1} V(H - z)^{-1}\|_{2p} \\ & \leq \| (H_0 - z)^{-1} \| \| G \mathbb{1}_{(-\infty, \lambda_1)}(H_0) \|_{2p} \| V_0 \| \| G(H - i)^{-1} \| \| (H - i)(H - z)^{-1} \| \\ & = O(|\operatorname{Im} z|^{-2}), \quad \operatorname{Im} z \rightarrow 0. \end{aligned}$$

Combining this with (3.8), we obtain that the integral in (3.9) converges absolutely in the norm of \mathbf{S}_{2p} . In view of (3.7), this yields the inclusion (3.4). The second inclusion (3.5) is proven by following exactly the same sequence of steps. \square

Proof of Lemma 2.3. We have

$$F_0(\lambda) = (G \mathbb{1}_{(-\infty, \lambda)}(H_0))(G \mathbb{1}_{(-\infty, \lambda)}(H_0))^*,$$

where, by the first inclusion in (3.6), both factors are in \mathbf{S}_{2p} . Using the Hölder inequality for Schatten classes, we obtain that $F_0(\lambda) \in \mathbf{S}_p$. Similarly, using the second inclusion in (3.6), we obtain $F(\lambda) \in \mathbf{S}_p$. Further, $\Pi_\varepsilon^{(1)}$ can be written as

$$\Pi_\varepsilon^{(1)} = (\mathbb{1}_{(-\infty, -\varepsilon)}(H_0) \mathbb{1}_{(\varepsilon, \infty)}(H)) (\mathbb{1}_{(\varepsilon, \infty)}(H) \mathbb{1}_{(-\infty, -\varepsilon)}(H_0)),$$

where both terms are in \mathbf{S}_{2p} by Lemma 3.2. Thus, $\Pi_\varepsilon^{(1)} \in \mathbf{S}_p$. Finally, consider $\Pi_\varepsilon^{(2)}$; by the definition of the functions ψ_ε^\pm , we have

$$\begin{aligned} \Pi_\varepsilon^{(2)} &= \psi_\varepsilon^-(H_0) \psi_\varepsilon^+(H) \psi_\varepsilon^-(H_0) \\ &= \psi_\varepsilon^-(H_0) \mathbb{1}_{(-\infty, -\varepsilon)}(H_0) \mathbb{1}_{(\varepsilon, \infty)}(H) \psi_\varepsilon^+(H) \mathbb{1}_{(\varepsilon, \infty)}(H) \mathbb{1}_{(-\infty, -\varepsilon)}(H_0) \psi_\varepsilon^-(H_0), \end{aligned}$$

and so the result again follows by Lemma 3.2. \square

4. SPECTRAL DENSITY OF A HANKEL OPERATOR

For $0 < \varepsilon \leq \delta$ let Γ_ε be the Hankel-type integral operator in $L^2(\mathbb{R}_+)$ with the integral kernel $\gamma_\varepsilon(s+t)$, $s, t \in \mathbb{R}_+$, where γ_ε is given by

$$\gamma_\varepsilon(t) = \int_\varepsilon^\delta e^{-t\lambda} d\lambda = \frac{e^{-t\varepsilon} - e^{-t\delta}}{t}, \quad t \in \mathbb{R}_+.$$

Lemma 4.1. *The operator Γ_ε belongs to the trace class and satisfies*

$$\Gamma_\varepsilon \geq 0, \quad \|\Gamma_\varepsilon\| \leq \pi. \quad (4.1)$$

Moreover, for any $q \geq 1$, one has

$$\lim_{\varepsilon \rightarrow +0} |\ln \varepsilon|^{-1} \operatorname{Tr} (\Gamma_\varepsilon)^q = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\pi}{\cosh(\pi x)} \right)^q dx. \quad (4.2)$$

Remark. In fact, one can check that $\Gamma_\varepsilon \in \mathbf{S}_q$ for all $q > 0$ and that (4.2) holds for all $q > 0$.

The proof of Lemma 4.1 relies on some well-known facts about the Carleman operator, that is, the Hankel operator in $L^2(\mathbb{R}_+)$ with the integral kernel $(t+s)^{-1}$, which we recall next. Let \mathcal{L} be the (self-adjoint) operator of the Laplace transform in $L^2(\mathbb{R}_+)$:

$$(\mathcal{L}f)(t) = \int_0^\infty e^{-xt} f(x) dx, \quad f \in L^2(\mathbb{R}_+). \quad (4.3)$$

Clearly, the Carleman operator can be written as \mathcal{L}^2 . This operator can be explicitly diagonalized. Namely, let $U : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R})$ be the unitary operator defined by

$$(Uf)(x) = e^{x/2} f(e^x), \quad x \in \mathbb{R}, \quad f \in L^2(\mathbb{R}_+).$$

By an explicit calculation (see e.g. [Pe, Section 10.2]), we obtain that $U\mathcal{L}^2U^*$ is the operator of convolution with the function $1/(2 \cosh(x/2))$. Computing the Fourier transform of this function,

$$\int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{2 \cosh(x/2)} dx = \frac{\pi}{\cosh(\pi\xi)} =: b(\xi), \quad (4.4)$$

we obtain

$$U\mathcal{L}^2U^* = b(D). \quad (4.5)$$

Here, the operator D (as well as the operator X , needed later) are the self-adjoint operators given by

$$(Xf)(x) = xf(x), \quad (Df)(x) = -i \frac{d}{dx} f(x) \quad \text{in } L^2(\mathbb{R}). \quad (4.6)$$

Since $\|b\|_{L^\infty} = \pi$, we note that (4.5) implies, in particular, that

$$\|\mathcal{L}\| = \sqrt{\pi}. \quad (4.7)$$

Proof of Lemma 4.1. Along with the operator X defined by (4.6), we will need its half-line version

$$(X_+ f)(x) = x f(x), \quad \text{in } L^2(\mathbb{R}_+).$$

In terms of the Laplace transform \mathcal{L} , our operator Γ_ε can be factorized as

$$\Gamma_\varepsilon = \mathcal{L} \mathbb{1}_{(\varepsilon, \delta)}(X_+) \mathcal{L} = (\mathbb{1}_{(\varepsilon, \delta)}(X_+) \mathcal{L})^* (\mathbb{1}_{(\varepsilon, \delta)}(X_+) \mathcal{L}). \quad (4.8)$$

This proves that $\Gamma_\varepsilon \geq 0$ and, since it is easy to check that $\mathbb{1}_{(\varepsilon, \delta)}(X_+) \mathcal{L} \in \mathbf{S}_2$, it follows that Γ_ε is trace class. Moreover, (4.7) implies $\|\Gamma_\varepsilon\| \leq \pi$.

Let us prove (4.2). Using the notation \approx introduced in Section 1.3, we deduce from (4.8) that $\Gamma_\varepsilon \approx \mathbb{1}_{(\varepsilon, \delta)}(X_+) \mathcal{L}^2 \mathbb{1}_{(\varepsilon, \delta)}(X_+)$. Thus, it follows from (4.5) that

$$\Gamma_\varepsilon \approx \mathbb{1}_{(\varepsilon, \delta)}(X_+) U^* b(D) U \mathbb{1}_{(\varepsilon, \delta)}(X_+) = U^* \mathbb{1}_{(\ln \varepsilon, \ln \delta)}(X) b(D) \mathbb{1}_{(\ln \varepsilon, \ln \delta)}(X) U.$$

From here we obtain

$$\mathrm{Tr} f(\Gamma_\varepsilon) = \mathrm{Tr} f(\mathbb{1}_{(\ln \varepsilon, \ln \delta)}(X) b(D) \mathbb{1}_{(\ln \varepsilon, \ln \delta)}(X)),$$

for any continuous function with $f(0) = 0$.

Now we first observe that for $q = 1$, formula (4.2) is a direct calculation of the trace of $\mathbb{1}_{(\ln \varepsilon, \ln \delta)}(X) b(D)$. For $q \geq 2$, we employ the following result of [LS]. Let P be an orthogonal projection in a Hilbert space and let B be a self-adjoint operator such that PB is Hilbert–Schmidt. Then for any $f \in C^2(\mathbb{R})$ with $f(0) = 0$, one has

$$|\mathrm{Tr} f(PBP) - \mathrm{Tr} P f(B) P| \leq \frac{1}{2} \|f''\|_{L^\infty} \|PB(1 - P)\|_2^2. \quad (4.9)$$

Let us take $P = \mathbb{1}_{(\ln \varepsilon, \ln \delta)}(X)$, $B = b(D)$, and $f(t) = t^q$, $q \geq 2$. Then

$$\mathrm{Tr} f(PBP) = \mathrm{Tr}(\Gamma_\varepsilon)^q, \quad (4.10)$$

and

$$\begin{aligned} \mathrm{Tr} P f(B) P &= \frac{1}{2\pi} \int_{\mathbb{R}} \mathbb{1}_{(\ln \varepsilon, \ln \delta)}(x) dx \int_{\mathbb{R}} b(\xi)^q d\xi \\ &= (|\ln \varepsilon| + O(1)) \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\pi}{\cosh(\pi x)} \right)^q dx. \end{aligned} \quad (4.11)$$

Thus, it remains to estimate the right side in (4.9):

$$\|PB(1 - P)\|_2^2 = \|(PB - BP)(1 - P)\|_2^2 \leq \|[P, B]\|_2^2.$$

Formula (4.4) implies that $[P, B]$ has integral kernel

$$\frac{\mathbb{1}_{(\ln \varepsilon, \ln \delta)}(x) - \mathbb{1}_{(\ln \varepsilon, \ln \delta)}(y)}{2 \cosh((x - y)/2)}, \quad x, y \in \mathbb{R}.$$

Thus,

$$\|[P, B]\|_2^2 = \iint_{\mathbb{R} \times \mathbb{R}} \frac{(\mathbb{1}_{(\ln \varepsilon, \ln \delta)}(x) - \mathbb{1}_{(\ln \varepsilon, \ln \delta)}(y))^2}{4 \cosh^2((x - y)/2)} dx dy = \int_{\mathbb{R}} \frac{\varphi(z)}{4 \cosh^2(z/2)} dz$$

with

$$\varphi(z) = \int_{\mathbb{R}} (\mathbb{1}_{(\ln \varepsilon, \ln \delta)}(y+z) - \mathbb{1}_{(\ln \varepsilon, \ln \delta)}(y))^2 dy = 2 \min\{|z|, \ln \delta - \ln \varepsilon\} \leq 2|z|.$$

We obtain $\| [P, B] \|_2^2 \leq \int_{\mathbb{R}} \frac{|z|}{2 \cosh^2(z/2)} dz < \infty$ uniformly in $\varepsilon > 0$. Returning to (4.9), we obtain

$$|\mathrm{Tr} f(PBP) - \mathrm{Tr} Pf(B)P| \leq C.$$

(Here we also used the fact that the L^∞ -norm of f'' needs only be evaluated on the finite interval $[0, \|B\|] = [0, \pi]$.) Combining this with (4.10), (4.11), we obtain the required statement for $f(t) = t^q$, $q \geq 2$.

Now assume that $f(t) = t^q$ with $1 < q < 2$ or, more generally, that $f(t) = tg(t)$ with g continuous on $[0, 1]$. Then, for any $\delta > 0$, there is a polynomial P with $\|P - g\|_\infty \leq \delta$. Let $f^{(1)}(t) = tP(t)$ and $f^{(2)}(t) = t(g(t) - P(t))$. According to the first part of the proof,

$$\lim_{\varepsilon \rightarrow 0} |\ln \varepsilon|^{-1} \mathrm{Tr} f^{(1)}(\Gamma_\varepsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f^{(1)}\left(\frac{\pi}{\cosh(\pi x)}\right) dx$$

and, by (4.11) with $q = 1$,

$$|\mathrm{Tr} f^{(2)}(\Gamma_\varepsilon)| \leq \delta \mathrm{Tr} \Gamma_\varepsilon = \delta(|\ln \varepsilon| + O(1)) \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\pi}{\cosh(\pi x)} dx \leq C_1 \delta |\ln \varepsilon|.$$

On the other hand,

$$\begin{aligned} & \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} f^{(1)}\left(\frac{\pi}{\cosh(\pi x)}\right) dx - \frac{1}{2\pi} \int_{-\infty}^{\infty} f\left(\frac{\pi}{\cosh(\pi x)}\right) dx \right| \\ & \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| f^{(2)}\left(\frac{\pi}{\cosh(\pi x)}\right) \right| dx \\ & \leq \frac{\delta}{2\pi} \int_{-\infty}^{\infty} \frac{\pi}{\cosh(\pi x)} dx = C_2 \delta. \end{aligned}$$

Thus,

$$\limsup_{\varepsilon \rightarrow 0} \left| |\ln \varepsilon|^{-1} \mathrm{Tr} f(\Gamma_\varepsilon) - \frac{1}{2\pi} \int_{-\infty}^{\infty} f\left(\frac{\pi}{\cosh(\pi x)}\right) dx \right| \leq (C_1 + C_2) \delta.$$

Since δ is arbitrary, we obtain the asymptotics for any f of the above form and, in particular, for $f(t) = t^q$, $1 < q < 2$. \square

Remark 4.2. (1) It is clear from the proof that if $q \geq 2$ or $q = 1$, then in fact we have a stronger statement:

$$\mathrm{Tr} (\Gamma_\varepsilon)^q = |\ln \varepsilon| \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\pi}{\cosh(\pi x)} \right)^q dx + O(1), \quad \varepsilon \rightarrow +0.$$

- (2) The crucial fact used in the proof above that Γ_ε is unitarily equivalent to the pseudo-differential operator

$$\mathbb{1}_{(\varepsilon, \delta)}(e^X)b(D)\mathbb{1}_{(\varepsilon, \delta)}(e^X) \quad \text{in } L^2(\mathbb{R}),$$

is a special case of a more general result of Yafaev [Ya3]; see also [W] for an older related result.

- (3) Note that the standard Berezin–Lieb inequality (see, e.g., [LS]) yields for arbitrary real numbers $q \geq 1$ the one-sided bound

$$\begin{aligned} \text{Tr}(\Gamma_\varepsilon)^q &\leq \frac{1}{2\pi} \int_{\mathbb{R}} \mathbb{1}_{(\ln \varepsilon, \ln \delta)}(x) dx \int_{-\infty}^{\infty} \left(\frac{\pi}{\cosh(\pi \xi)} \right)^q d\xi \\ &= \frac{1}{2\pi} (|\ln \varepsilon| + O(1)) \int_{-\infty}^{\infty} \left(\frac{\pi}{\cosh(\pi \xi)} \right)^q d\xi. \end{aligned}$$

Thus the approximation argument above was only needed for a lower bound for $1 < q < 2$.

5. ESTIMATES FOR OPERATOR-VALUED HANKEL OPERATORS

We need some Schatten class estimates for Hankel operators acting in $L^2(\mathbb{R}_+, \mathcal{K})$. Fix $q \geq 1$; let $\sigma : \mathbb{R}_+ \rightarrow \mathbf{S}_q(\mathcal{K})$ be a measurable function and let K be the integral Hankel operator in $L^2(\mathbb{R}_+, \mathcal{K})$ (see (2.16)) with the kernel $k = k(t)$, given by the Laplace transform of σ :

$$k(t) = \int_0^\infty e^{-\lambda t} \sigma(\lambda) d\lambda.$$

Lemma 5.1. *For $1 \leq q < \infty$, one has*

$$\|K\|_q^q \leq \pi^{q-1} \int_0^\infty \|\sigma(\lambda)\|_q^q \frac{d\lambda}{2\lambda}$$

and

$$\|K\| \leq \pi \operatorname{ess\,sup}_{\lambda > 0} \|\sigma(\lambda)\|.$$

Remark. In the scalar case $\mathcal{K} = \mathbb{C}$, this result has appeared in [W] for $q = \infty$ and in [H] for $q = 1$.

Proof. We can write $K = \mathcal{L}\sigma\mathcal{L}$, where \mathcal{L} denotes, as in (4.3), the Laplace transform. Then, for $q = \infty$, we have

$$\|K\| \leq \|\mathcal{L}\|^2 \|\sigma\| = \pi \operatorname{ess\,sup}_{\lambda > 0} \|\sigma(\lambda)\|, \quad (5.1)$$

where we have used (4.7). For $q = 1$, we have

$$\|K\|_1 \leq \|\mathcal{L}|\sigma|^{1/2}\|_2^2 = \int_0^\infty \| |\sigma(\lambda)|^{1/2} \|_2^2 \frac{d\lambda}{2\lambda} = \int_0^\infty \|\sigma(\lambda)\|_1 \frac{d\lambda}{2\lambda}. \quad (5.2)$$

For $1 < q < \infty$, the bound follows by complex interpolation. For the sake of completeness we include the details of this argument. For fixed $1 < q < \infty$ we consider the analytic family of operators

$$K_z = \mathcal{L}u|\sigma|^{zq}\mathcal{L},$$

where, for $\lambda > 0$, $u(\lambda)$ is a partial isometry in \mathcal{K} such that $\sigma(\lambda) = u(\lambda)|\sigma(\lambda)|$. The bounds (5.1) and (5.2) show that $K_z \in \mathbf{B}$ (= the class of bounded operators) if $\operatorname{Re} z = 0$ with

$$\|K_z\| \leq \pi,$$

and that $K_z \in \mathbf{S}_1$ if $\operatorname{Re} z = 1$ with

$$\|K_z\|_1 \leq \int_0^\infty \| |\sigma(\lambda)|^{zq} \|_1 \frac{d\lambda}{2\lambda} = \int_0^\infty \|\sigma(\lambda)\|_q^q \frac{d\lambda}{2\lambda}.$$

Thus, by complex interpolation (see, e.g., [S, Thm. 2.9]), $K_{1/q} \in \mathbf{S}_q$ with

$$\|K_{1/q}\|_q \leq \pi^{1-1/q} \left(\int_0^\infty \|\sigma(\lambda)\|_q^q \frac{d\lambda}{2\lambda} \right)^{1/q}.$$

Since $K_{1/q} = \mathcal{L}\sigma\mathcal{L} = K$, this proves the lemma. \square

6. THE OPERATORS \mathcal{Z}_ε AND $\mathcal{Z}_\varepsilon^{(0)}$

Let $\mathcal{Z}_\varepsilon, \mathcal{Z}_\varepsilon^{(0)} : L^2(\mathbb{R}_+, \mathcal{K}) \rightarrow \mathcal{H}$ be the operators defined by (2.14), (2.15), and let $K_\varepsilon = (\mathcal{Z}_\varepsilon)^* \mathcal{Z}_\varepsilon$, $K_\varepsilon^{(0)} = (\mathcal{Z}_\varepsilon^{(0)})^* \mathcal{Z}_\varepsilon^{(0)}$. An inspection shows that K_ε and $K_\varepsilon^{(0)}$ are Hankel operators in $L^2(\mathbb{R}_+, \mathcal{K})$ with the kernels given by

$$k_\varepsilon(t) = V_0 G e^{-tH} \mathbb{1}_{(\varepsilon, \delta)}(H) G^* V_0, \quad k_\varepsilon^{(0)}(t) = G e^{tH_0} \mathbb{1}_{(-\delta, -\varepsilon)}(H) G^*, \quad t > 0.$$

First we check that these operators are bounded and give an estimate for their norms as $\varepsilon \rightarrow 0$:

Lemma 6.1. *We have*

$$\sup_{0 < \varepsilon \leq \delta} (\|\mathcal{Z}_\varepsilon\| + \|\mathcal{Z}_\varepsilon^{(0)}\|) < \infty \tag{6.1}$$

and

$$\|(\mathcal{Z}_\varepsilon)^* \mathcal{Z}_\varepsilon\|_p^p + \|(\mathcal{Z}_\varepsilon^{(0)})^* \mathcal{Z}_\varepsilon^{(0)}\|_p^p = O(|\ln \varepsilon|) \quad \text{as } \varepsilon \rightarrow 0.$$

The bound (6.1) is already contained in [Pu], but we include a proof for the sake of completeness.

Proof. By the spectral theorem, we have

$$k_\varepsilon(t) = V_0 G e^{-tH} \mathbb{1}_{(\varepsilon, \delta)}(H) G^* V_0 = \int_\varepsilon^\delta e^{-t\lambda} dF(\lambda) = \int_\varepsilon^\delta e^{-t\lambda} F'(\lambda) d\lambda. \tag{6.2}$$

Thus, the kernel k_ε is a Laplace transform of an operator valued measure and so we can apply Lemma 5.1. From Assumption 2.4 we know that

$$\sup_{0 \leq \lambda \leq \delta} \|F'(\lambda)\| \leq \sup_{0 \leq \lambda \leq \delta} \|F'(\lambda)\|_p < \infty,$$

and therefore, by Lemma 5.1 with $q = \infty$, we get the uniform boundedness of $\|\mathcal{Z}_\varepsilon\|$. Further, again by Lemma 5.1 with $q = p$,

$$\|(\mathcal{Z}_\varepsilon)^* \mathcal{Z}_\varepsilon\|_p^p \leq \pi^{p-1} \int_\varepsilon^\delta \|F'(\lambda)\|_p^p \frac{d\lambda}{2\lambda} \leq C \int_\varepsilon^\delta \frac{d\lambda}{\lambda} = O(|\ln \varepsilon|) \quad \text{as } \varepsilon \rightarrow 0.$$

This proves the lemma for \mathcal{Z}_ε . The proof for $\mathcal{Z}_\varepsilon^{(0)}$ is similar and involves the representation

$$k_\varepsilon^{(0)}(t) = \int_{-\delta}^{-\varepsilon} e^{t\lambda} F'_0(t) dt = \int_\varepsilon^\delta e^{-t\lambda} F'_0(-t) dt. \quad (6.3)$$

□

Lemma 6.2. *The factorisation (2.13) holds true:*

$$\mathbb{1}_{(\varepsilon, \delta)}(H) \mathbb{1}_{(-\delta, -\varepsilon)}(H_0) = \mathcal{Z}_\varepsilon (\mathcal{Z}_\varepsilon^{(0)})^*.$$

Proof. This is a calculation from [Pu], which we reproduce for completeness. Let

$$L(t) = \mathbb{1}_{(\varepsilon, \delta)}(H) e^{-tH} e^{tH_0} \mathbb{1}_{(-\delta, -\varepsilon)}(H_0).$$

Then we have $L(0) = \mathbb{1}_{(\varepsilon, \delta)}(H) \mathbb{1}_{(-\delta, -\varepsilon)}(H_0)$, $L(+\infty) = 0$, and

$$\begin{aligned} L'(t) &= -\mathbb{1}_{(\varepsilon, \delta)}(H) e^{-tH} V e^{tH_0} \mathbb{1}_{(-\delta, -\varepsilon)}(H_0) \\ &= -(V_0 G e^{-tH} \mathbb{1}_{(\varepsilon, \delta)}(H))^* (G e^{tH_0} \mathbb{1}_{(-\delta, -\varepsilon)}(H_0)), \quad t > 0. \end{aligned}$$

Substituting this into

$$L(0) - L(+\infty) = - \int_0^\infty L'(t) dt,$$

and recalling the definition of the operators \mathcal{Z}_ε , $\mathcal{Z}_\varepsilon^{(0)}$, we obtain the required identity. □

In the next lemma we shall determine the leading order behavior of the Hankel operators $(\mathcal{Z}_\varepsilon^{(0)})^* \mathcal{Z}_\varepsilon^{(0)}$ and $(\mathcal{Z}_\varepsilon)^* \mathcal{Z}_\varepsilon$ in $L^2(\mathbb{R}_+, \mathcal{K})$ as $\varepsilon \rightarrow 0$. It turns out that these operators can be approximated in \mathbf{S}_p norm by Hankel operators with the explicit kernels

$$\gamma_\varepsilon(t) F'_0(0), \quad \gamma_\varepsilon(t) F'(0), \quad t > 0, \quad (6.4)$$

where γ_ε is the model kernel considered in Section 4,

$$\gamma_\varepsilon(t) = \int_\varepsilon^\delta e^{-t\lambda} d\lambda.$$

Identifying $L^2(\mathbb{R}_+, \mathcal{K})$ with $L^2(\mathbb{R}_+) \otimes \mathcal{K}$, we shall denote the Hankel operators with the kernels (6.4) by

$$\Gamma_\varepsilon \otimes F'_0(0), \quad \Gamma_\varepsilon \otimes F'(0).$$

Lemma 6.3. *We have*

$$\|(\mathcal{Z}_\varepsilon^{(0)})^* \mathcal{Z}_\varepsilon^{(0)} - \Gamma_\varepsilon \otimes F'_0(0)\|_p = O(1) \quad \text{as } \varepsilon \rightarrow 0, \quad (6.5)$$

$$\|(\mathcal{Z}_\varepsilon)^* \mathcal{Z}_\varepsilon - \Gamma_\varepsilon \otimes F'(0)\|_p = O(1) \quad \text{as } \varepsilon \rightarrow 0. \quad (6.6)$$

Proof. Let us first prove (6.6). Recalling formula (6.2) for k_ε , we see that the Hankel operator $(\mathcal{Z}_\varepsilon)^* \mathcal{Z}_\varepsilon - \Gamma_\varepsilon \otimes F'(0)$ has the kernel

$$k_\varepsilon(t) - \gamma_\varepsilon(t)F'(0) = \int_\varepsilon^\delta e^{-t\lambda}(F'(\lambda) - F'(0))d\lambda.$$

Applying Lemma 5.1 with $q = p$, we get

$$\|(\mathcal{Z}_\varepsilon)^* \mathcal{Z}_\varepsilon - \Gamma_\varepsilon \otimes F'(0)\|_p^p \leq \pi^{p-1} \int_\varepsilon^\delta \|F'(\lambda) - F'(0)\|_p^p \frac{d\lambda}{2\lambda}. \quad (6.7)$$

By Assumption 2.4, we have $\|F'(\lambda) - F'(0)\|_p = O(\lambda^\varkappa)$, $\lambda \rightarrow 0$, with some $\varkappa > 0$. It follows that the right side in (6.7) is bounded uniformly in $\varepsilon > 0$. This proves (6.6). The argument for (6.5) is similar and involves the representation (6.3) for $k_\varepsilon^{(0)}$. \square

Remark. Note that Lemma 6.3 together with Lemma 4.1 implies all the assertions in Lemma 6.1. We have chosen to prove Lemma 6.1 separately for pedagogic reasons, since it does not rely on the machinery to prove Lemma 4.1.

7. SPECTRAL LOCALIZATION

Let $\Pi_\varepsilon^{(1)}$ and $\tilde{\Pi}_\varepsilon^{(1)}$ be the operators defined by (2.11), (2.12). In this section, we prove

Lemma 7.1. *Let Assumptions 2.2 and 2.4 hold true. Then for all $q \geq p$ one has*

$$\|\tilde{\Pi}_\varepsilon^{(1)} - \Pi_\varepsilon^{(1)}\|_q^q = O(|\ln \varepsilon|^{1/2}) \quad \text{as } \varepsilon \rightarrow 0. \quad (7.1)$$

Proof. Setting

$$P_\varepsilon = \mathbb{1}_{(\varepsilon, \infty)}(H) \mathbb{1}_{(-\infty, -\varepsilon)}(H_0), \quad \tilde{P}_\varepsilon = \mathbb{1}_{(\varepsilon, \delta)}(H) \mathbb{1}_{(-\delta, -\varepsilon)}(H_0),$$

we can write

$$\Pi_\varepsilon^{(1)} = P_\varepsilon^* P_\varepsilon, \quad \tilde{\Pi}_\varepsilon^{(1)} = \tilde{P}_\varepsilon^* \tilde{P}_\varepsilon.$$

First let us estimate the difference $P_\varepsilon - \tilde{P}_\varepsilon$. We have

$$\begin{aligned} P_\varepsilon - \tilde{P}_\varepsilon &= (\mathbb{1}_{(\varepsilon, \infty)}(H) \mathbb{1}_{(-\infty, -\varepsilon)}(H_0) - \mathbb{1}_{(\varepsilon, \delta)}(H) \mathbb{1}_{(-\infty, -\varepsilon)}(H_0)) \\ &\quad + (\mathbb{1}_{(\varepsilon, \delta)}(H) \mathbb{1}_{(-\infty, -\varepsilon)}(H_0) - \mathbb{1}_{(\varepsilon, \delta)}(H) \mathbb{1}_{(-\delta, -\varepsilon)}(H_0)) \\ &= \mathbb{1}_{[\delta, \infty)}(H) \mathbb{1}_{(-\infty, -\varepsilon)}(H_0) - \mathbb{1}_{(\varepsilon, \delta)}(H) \mathbb{1}_{(-\infty, -\delta]}(H_0). \end{aligned} \quad (7.2)$$

Using Lemma 3.2, we can estimate separately each of the two terms in the right side of (7.2):

$$\begin{aligned}\|\mathbb{1}_{[\delta,\infty)}(H)\mathbb{1}_{(-\infty,-\varepsilon)}(H_0)\|_{2p} &\leq \|\mathbb{1}_{[\delta,\infty)}(H)\mathbb{1}_{(-\infty,0)}(H_0)\|_{2p} < \infty, \\ \|\mathbb{1}_{(\varepsilon,\delta)}(H)\mathbb{1}_{(-\infty,-\delta]}(H_0)\|_{2p} &\leq \|\mathbb{1}_{(0,\delta)}(H)\mathbb{1}_{(-\infty,-\delta]}(H_0)\|_{2p} < \infty.\end{aligned}$$

It follows that

$$\|P_\varepsilon - \tilde{P}_\varepsilon\|_{2p} = O(1) \quad \text{as } \varepsilon \rightarrow 0. \quad (7.3)$$

Next, let us estimate \tilde{P}_ε . Since $\|\tilde{P}_\varepsilon\| \leq 1$, we have

$$\|\tilde{P}_\varepsilon\|_{2p}^{2p} = \| |\tilde{P}_\varepsilon|^{2p} \|_1 \leq \| |\tilde{P}_\varepsilon|^p \|_1 = \|\tilde{P}_\varepsilon\|_p^p. \quad (7.4)$$

Recall that by (2.13), we have

$$\tilde{P}_\varepsilon = \mathcal{Z}_\varepsilon(\mathcal{Z}_\varepsilon^{(0)})^*.$$

Thus, by Lemma 6.1,

$$\|\tilde{P}_\varepsilon\|_p^p = \|\mathcal{Z}_\varepsilon(\mathcal{Z}_\varepsilon^{(0)})^*\|_p^p \leq \|\mathcal{Z}_\varepsilon\|_{2p}^p \|\mathcal{Z}_\varepsilon^{(0)}\|_{2p}^p = O(|\ln \varepsilon|) \quad \text{as } \varepsilon \rightarrow 0.$$

Combining these formulas, we obtain

$$\|\tilde{P}_\varepsilon\|_{2p} = O(|\ln \varepsilon|^{1/2p}) \quad \text{as } \varepsilon \rightarrow 0. \quad (7.5)$$

Combining (7.3) and (7.5), we also obtain

$$\|P_\varepsilon\|_{2p} = O(|\ln \varepsilon|^{1/2p}) \quad \text{as } \varepsilon \rightarrow 0. \quad (7.6)$$

Let us prove (7.1) for $q = p$. We have

$$\Pi_\varepsilon^{(1)} - \tilde{\Pi}_\varepsilon^{(1)} = P_\varepsilon^* P_\varepsilon - \tilde{P}_\varepsilon^* \tilde{P}_\varepsilon = (P_\varepsilon^* - \tilde{P}_\varepsilon^*) P_\varepsilon + \tilde{P}_\varepsilon^* (P_\varepsilon - \tilde{P}_\varepsilon),$$

and therefore, by (7.3), (7.5), (7.6),

$$\|\Pi_\varepsilon^{(1)} - \tilde{\Pi}_\varepsilon^{(1)}\|_p \leq \|P_\varepsilon^* - \tilde{P}_\varepsilon^*\|_{2p} \|P_\varepsilon\|_{2p} + \|\tilde{P}_\varepsilon^*\|_{2p} \|P_\varepsilon - \tilde{P}_\varepsilon\|_{2p} = O(|\log \varepsilon|^{1/2p}),$$

as required. In order to derive (7.1) for $q > p$, we use the fact that $\|\tilde{\Pi}_\varepsilon^{(1)} - \Pi_\varepsilon^{(1)}\| \leq 2$ and argue as in (7.4):

$$\|\tilde{\Pi}_\varepsilon^{(1)} - \Pi_\varepsilon^{(1)}\|_q^q = \| |\tilde{\Pi}_\varepsilon^{(1)} - \Pi_\varepsilon^{(1)}|^q \|_1 \leq 2^{q-p} \| |\tilde{\Pi}_\varepsilon^{(1)} - \Pi_\varepsilon^{(1)}|^p \|_1 = 2^{q-p} \|\tilde{\Pi}_\varepsilon^{(1)} - \Pi_\varepsilon^{(1)}\|_p^p.$$

□

Remark 7.2. Note that (7.3) immediately implies

$$\|\Pi_\varepsilon^{(1)} - \tilde{\Pi}_\varepsilon^{(1)}\|_{2p} = O(1),$$

which suffices for the proof of Theorem 2.5 for $f(t) = t^n$ with $n \geq 2p$.

8. PUTTING IT ALL TOGETHER

8.1. Auxiliary statements. First, we need to relate the scattering matrix to the operators F'_0, F' . Note that $F_0(\lambda)$ and $F(\lambda)$ are non-decreasing with respect to λ and so $F'_0(\lambda)$ and $F'(\lambda)$ are non-negative. In particular, $F'_0(\lambda)^{1/2}$ and $F'(\lambda)^{1/2}$ are well-defined.

Below we frequently use the notation \approx introduced in Section 1.3.

Lemma 8.1. *Let Assumptions 2.2, 2.4 hold true; then*

$$\frac{1}{4}|S(0) - I|^2 \approx \pi^2 F'(0)^{1/2} F'_0(0) F'(0)^{1/2} \quad (8.1)$$

and

$$\sum_{\ell=1}^L a_\ell^p < \infty. \quad (8.2)$$

Proof. This is essentially a known statement (see e.g. [Pu, Lemma 4] or [GKMO, Corollary 4.31]). For completeness, we briefly recall the proof. First note that by unitarity of $S(0)$ we have

$$\frac{1}{4}|S(0) - I|^2 = \frac{1}{4}(S(0)^* - I)(S(0) - I) = \frac{1}{2} \operatorname{Re}(I - S(0)).$$

Next, by the stationary representation for the scattering matrix (see e.g. [Ya1, Theorem 5.5.4]), the operator $S(0)$ is unitarily equivalent to the operator (recall that $T(z)$ is defined in (2.3))

$$\tilde{S}(0) = I - 2\pi i F'_0(0)^{1/2} (V_0 - T(+i0)) F'_0(0)^{1/2} \quad \text{in } \mathcal{K}.$$

It follows that

$$\begin{aligned} \frac{1}{2} \operatorname{Re}(I - S(0)) &\approx \frac{1}{2} \operatorname{Re}(I - \tilde{S}(0)) \\ &= \pi \operatorname{Im}(F'_0(0)^{1/2} T(+i0) F'_0(0)^{1/2}) = \pi^2 F'_0(0)^{1/2} F'(0) F'_0(0)^{1/2}, \end{aligned}$$

where we have used (2.9) at the last step. Denoting $X = F'(0)^{1/2} F'_0(0)^{1/2}$, the last operator can be transformed as

$$\pi^2 F'_0(0)^{1/2} F'(0) F'_0(0)^{1/2} = \pi^2 X^* X \approx \pi^2 X X^* = \pi^2 F'(0)^{1/2} F'_0(0) F'(0)^{1/2},$$

which yields (8.1). By Assumption 2.4, the operator in the right side of (8.1) is in the class $\mathbf{S}_{p/2}$. The relation (8.1) implies that the non-zero a_ℓ^2 coincide with the non-zero eigenvalues of this operator; thus, we obtain (8.2). \square

Lemma 8.2. *Let $X_\varepsilon, Y_\varepsilon$ be non-negative, compact operators depending on $\varepsilon > 0$. Assume that for some $q \geq 1$, we have*

$$\|X_\varepsilon\|_q^q = O(|\ln \varepsilon|), \quad \|X_\varepsilon - Y_\varepsilon\|_q^q = o(|\ln \varepsilon|) \quad \text{as } \varepsilon \rightarrow 0. \quad (8.3)$$

Then

$$\operatorname{Tr} Y_\varepsilon^q - \operatorname{Tr} X_\varepsilon^q = o(|\ln \varepsilon|), \quad \varepsilon \rightarrow 0.$$

Of course, we choose the function $|\ln \varepsilon|$ here simply because this is what comes up in our proof in the next subsection.

Proof. We note that for $X \geq 0$, we have $\text{Tr } X^q = \|X\|_q^q$. Now the statement of the lemma follows directly from the estimate

$$\|Y\|_q^q - \|X\|_q^q \leq q \max\{\|Y\|_q^{q-1}, \|X\|_q^{q-1}\} \|Y - X\|_q.$$

To prove the latter estimate, it suffices to use the elementary inequality

$$|b^q - a^q| \leq q \max\{b^{q-1}, a^{q-1}\} |b - a|, \quad a \geq 0, \quad b \geq 0,$$

with $a = \|X\|_q$, $b = \|Y\|_q$ and the inverse triangle inequality $|\|Y\|_q - \|X\|_q| \leq \|Y - X\|_q$. \square

8.2. The case $f(t) = t^q$.

Lemma 8.3. *For any $q \geq p$, Theorem 2.5 holds true with $f(t) = t^q$. That is,*

$$\lim_{\varepsilon \rightarrow +0} |\ln \varepsilon|^{-1} \text{Tr} (\Pi_\varepsilon^{(1)})^q = \frac{1}{2\pi} \sum_{\ell=1}^L a_\ell^{2q} \int_{-\infty}^{\infty} \frac{dx}{\cosh^{2q}(\pi x)}. \quad (8.4)$$

Proof. Let us denote the operator on the right side of (8.1) by A ,

$$A = \pi^2 F'(0)^{1/2} F_0'(0) F'(0)^{1/2} \quad \text{in } \mathcal{K}. \quad (8.5)$$

It follows from Lemma 8.1 that $\{a_\ell^2\}_{\ell=1}^L$ are the non-zero eigenvalues of A . In the course of the proof we progressively reduce the problem for $\Pi_\varepsilon^{(1)}$ to the problem for the following operators:

$$\tilde{\Pi}_\varepsilon^{(1)} = \mathcal{Z}_\varepsilon^{(0)} (\mathcal{Z}_\varepsilon)^* \mathcal{Z}_\varepsilon (\mathcal{Z}_\varepsilon^{(0)})^*, \quad (8.6)$$

$$M_{1,\varepsilon} = \mathcal{Z}_\varepsilon^{(0)} (\Gamma_\varepsilon \otimes F'(0)) (\mathcal{Z}_\varepsilon^{(0)})^*, \quad (8.7)$$

$$M_{2,\varepsilon} = (\Gamma_\varepsilon \otimes F'(0))^{1/2} (\mathcal{Z}_\varepsilon^{(0)})^* \mathcal{Z}_\varepsilon^{(0)} (\Gamma_\varepsilon \otimes F'(0))^{1/2}, \quad (8.8)$$

$$\pi^{-2} \Gamma_\varepsilon^2 \otimes A = (\Gamma_\varepsilon \otimes F'(0))^{1/2} (\Gamma_\varepsilon \otimes F_0'(0)) (\Gamma_\varepsilon \otimes F'(0))^{1/2}. \quad (8.9)$$

Here formula (8.6) follows from the factorization (2.13), formulas (8.7) and (8.8) are the definitions of the auxiliary operators $M_{1,\varepsilon}$ and $M_{2,\varepsilon}$, and formula (8.9) follows from the definition (8.5) of A . It is convenient to start from the bottom operator (8.9) and to move up.

Denote the right side of (8.4) by Δ_q . By Lemma 4.1, we have

$$\text{Tr}(\pi^{-2} \Gamma_\varepsilon^2 \otimes A)^q = \sum_{\ell=1}^L a_\ell^{2q} \text{Tr}(\pi^{-2} \Gamma_\varepsilon^2)^q = |\ln \varepsilon| \Delta_q + o(|\ln \varepsilon|). \quad (8.10)$$

Let us estimate the difference

$$M_{2,\varepsilon} - \pi^{-2} \Gamma_\varepsilon^2 \otimes A = (\Gamma_\varepsilon \otimes F'(0))^{1/2} ((\mathcal{Z}_\varepsilon^{(0)})^* \mathcal{Z}_\varepsilon^{(0)} - \Gamma_\varepsilon \otimes F_0'(0)) (\Gamma_\varepsilon \otimes F'(0))^{1/2}. \quad (8.11)$$

By Lemma 4.1 and Lemma 6.3,

$$\begin{aligned}
\|M_{2,\varepsilon} - \pi^{-2}\Gamma_\varepsilon^2 \otimes A\|_q &\leq \|M_{2,\varepsilon} - \pi^{-2}\Gamma_\varepsilon^2 \otimes A\|_p \\
&\leq \|(\Gamma_\varepsilon \otimes F'(0))^{1/2}\|^2 \left\| (\mathcal{Z}_\varepsilon^{(0)})^* \mathcal{Z}_\varepsilon^{(0)} - \Gamma_\varepsilon \otimes F'_0(0) \right\|_p \\
&= \|\Gamma_\varepsilon\|^2 \|F'(0)\| \left\| (\mathcal{Z}_\varepsilon^{(0)})^* \mathcal{Z}_\varepsilon^{(0)} - \Gamma_\varepsilon \otimes F'_0(0) \right\|_p = O(1).
\end{aligned} \tag{8.12}$$

Let us apply Lemma 8.2 with $X_\varepsilon = \pi^{-2}\Gamma_\varepsilon^2 \otimes A$ and $Y_\varepsilon = M_{2,\varepsilon}$. In the hypothesis (8.3) of this lemma, the first estimate follows from (8.10) and the second estimate holds by (8.12). We obtain

$$\lim_{\varepsilon \rightarrow 0+} |\ln \varepsilon|^{-1} \operatorname{Tr} (M_{2,\varepsilon})^q = \Delta_q. \tag{8.13}$$

Next, by definitions (8.7) and (8.8), we have $M_{2,\varepsilon} \approx M_{1,\varepsilon}$, and therefore (8.13) yields

$$\lim_{\varepsilon \rightarrow 0+} |\ln \varepsilon|^{-1} \operatorname{Tr} (M_{1,\varepsilon})^q = \Delta_q.$$

Further, similarly to (8.11), (8.12),

$$\widetilde{\Pi}_\varepsilon^{(1)} - M_{1,\varepsilon} = \mathcal{Z}_\varepsilon^{(0)} (\mathcal{Z}_\varepsilon^* \mathcal{Z}_\varepsilon - \Gamma_\varepsilon \otimes F'(0)) (\mathcal{Z}_\varepsilon^{(0)})^*,$$

and so by Lemmas 6.1 and 6.3

$$\|\widetilde{\Pi}_\varepsilon^{(1)} - M_{1,\varepsilon}\|_q \leq \|\widetilde{\Pi}_\varepsilon^{(1)} - M_{1,\varepsilon}\|_p \leq \|\mathcal{Z}_\varepsilon^{(0)}\|^2 \|\mathcal{Z}_\varepsilon^* \mathcal{Z}_\varepsilon - \Gamma_\varepsilon \otimes F'(0)\|_p = O(1).$$

Now we can apply Lemma 8.2 with $X_\varepsilon = M_{1,\varepsilon}$ and $Y_\varepsilon = \widetilde{\Pi}_\varepsilon^{(1)}$, which yields

$$\lim_{\varepsilon \rightarrow 0+} |\ln \varepsilon|^{-1} \operatorname{Tr} (\widetilde{\Pi}_\varepsilon^{(1)})^q = \Delta_q.$$

Finally, we apply Lemma 8.2 once again with $X_\varepsilon = \widetilde{\Pi}_\varepsilon^{(1)}$ and $Y_\varepsilon = \Pi_\varepsilon^{(1)}$. The second estimate in the hypothesis (8.3) is given by Lemma 7.1. This yields

$$\lim_{\varepsilon \rightarrow 0+} |\ln \varepsilon|^{-1} \operatorname{Tr} (\Pi_\varepsilon^{(1)})^q = \Delta_q,$$

as required. \square

8.3. Proofs of Theorem 2.5 and Corollary 2.6.

Proof of Theorem 2.5. First consider the operator $\Pi_\varepsilon^{(1)}$. By Lemma 8.3, the asymptotics (4.1) holds if f is t^q times a polynomial. The proof for a general f follows from the Weierstrass approximation theorem as at the end of the proof of Lemma 4.1.

Next, consider the operator $\Pi_\varepsilon^{(2)}$. By our assumptions on the functions ψ_ε^\pm , we have

$$\begin{aligned}
\Pi_\varepsilon^{(2)} &= \psi_\varepsilon^-(H_0) \psi_\varepsilon^+(H) \psi_\varepsilon^-(H_0) \leq \psi_\varepsilon^-(H_0) \mathbb{1}_{(\varepsilon, \infty)}(H) \psi_\varepsilon^-(H_0) \\
&\approx \mathbb{1}_{(\varepsilon, \infty)}(H) (\psi_\varepsilon^-(H_0))^2 \mathbb{1}_{(\varepsilon, \infty)}(H) \leq \mathbb{1}_{(\varepsilon, \infty)}(H) \mathbb{1}_{(-\infty, -\varepsilon)}(H_0) \mathbb{1}_{(\varepsilon, \infty)}(H) \approx \Pi_\varepsilon^{(1)}.
\end{aligned}$$

By the min-max principle, it follows that

$$\mathrm{Tr}(\Pi_\varepsilon^{(2)})^q \leq \mathrm{Tr}(\Pi_\varepsilon^{(1)})^q, \quad q \geq p. \quad (8.14)$$

Similarly, we have a lower bound

$$\begin{aligned} \Pi_\varepsilon^{(2)} &= \psi_\varepsilon^-(H_0) \psi_\varepsilon^+(H) \psi_\varepsilon^-(H_0) \geq \psi_\varepsilon^-(H_0) \mathbb{1}_{(2\varepsilon, \infty)}(H) \psi_\varepsilon^-(H_0) \\ &\approx \mathbb{1}_{(2\varepsilon, \infty)}(H) (\psi_\varepsilon^-(H_0))^2 \mathbb{1}_{(2\varepsilon, \infty)}(H) \leq \mathbb{1}_{(2\varepsilon, \infty)}(H) \mathbb{1}_{(-\infty, -2\varepsilon)}(H_0) \mathbb{1}_{(2\varepsilon, \infty)}(H) \approx \Pi_{2\varepsilon}^{(1)}, \end{aligned}$$

and therefore

$$\mathrm{Tr}(\Pi_\varepsilon^{(2)})^q \geq \mathrm{Tr}(\Pi_{2\varepsilon}^{(1)})^q, \quad q \geq p. \quad (8.15)$$

A combination (8.14) and (8.15) gives the analogue of Lemma 8.3 for $\Pi_\varepsilon^{(2)}$ and so again we obtain the required statement by application of the Weierstrass approximation theorem. \square

Proof of Corollary 2.6. Let $f(x)$ be a continuous function on $[0, 1]$ such that f vanishes in a neighbourhood of zero and $\ln(1 - x) \leq f(x)$. We have

$$\ln \det(I - \Pi_\varepsilon^{(j)}) = \mathrm{Tr} \ln(I - \Pi_\varepsilon^{(j)}) \leq \mathrm{Tr} f(\Pi_\varepsilon^{(j)}),$$

and therefore by Theorem 2.5,

$$\limsup_{\varepsilon \rightarrow 0+} |\ln \varepsilon|^{-1} \ln \det(I - \Pi_\varepsilon^{(j)}) \leq \frac{1}{2\pi} \sum_{\ell=1}^L \int_{-\infty}^{\infty} f\left(\frac{a_\ell^2}{\cosh^2(\pi x)}\right) dx.$$

Taking the infimum over all such f in the right side, we obtain the required statement. \square

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REFERENCES

- [DS] M. Dimassi, J. Sjöstrand, *Spectral Asymptotics in the Semi-Classical Limit*, Cambridge University Press, 1999.
- [GKM] M. Gebert, H. Küttler, P. Müller, *Anderson's orthogonality catastrophe*, Commun. Math. Phys. **329** (2014), no. 3, 979–998.
- [GKMO] M. Gebert, H. Küttler, P. Müller, P. Otte, *The exponent in the orthogonality catastrophe for Fermi gases*. Preprint (2014), arXiv:1407.2512.
- [GR] I. S. Gradshteyn, I. M. Ryzhik, *Table of integrals, series, and products*. Seventh edition. Elsevier/Academic Press, Amsterdam, 2007.
- [H] J. S. Howland, *Trace class Hankel operators*, Quart. J. Math. Oxford (2), **22** (1971), 147–159.
- [KOS] H. Küttler, P. Otte, W. Spitzer, *Anderson's orthogonality catastrophe for one-dimensional systems*. Ann. H. Poincaré, **15** (2013), no. 9, 1655–1696.
- [LS] A. Laptev, Yu. Safarov, *Szegő type limit theorems*. J. Funct. Anal. **138** (1996), no. 2, 544–559.
- [Pe] V. V. Peller, *Hankel operators and their applications*, Springer, 2003.

- [Pu] A. Pushnitski, *The scattering matrix and the differences of spectral projections*. Bull. Lond. Math. Soc. **40** (2008), no. 2, 227–238.
- [PYa] A. Pushnitski, D. Yafaev, *Spectral theory of discontinuous functions of self-adjoint operators and scattering theory*. J. Funct. Anal. **259** (2010), no. 8, 1950–1973.
- [RS] M. Reed, B. Simon, *Methods of modern mathematical physics. III: Scattering theory*. Academic Press, 1979.
- [S] B. Simon, *Trace ideals and their applications*. Second edition. Mathematical Surveys and Monographs **120**. Amer. Math. Soc., Providence, RI, 2005.
- [W] H. Widom, *Hankel matrices*, Trans. Amer. Math. Soc. **121**, (1966), no. 1, 1–35.
- [Ya1] D. Yafaev, *Mathematical Scattering Theory: General Theory*. AMS, 1992.
- [Ya2] D. Yafaev, *Mathematical Scattering Theory. Analytic Theory*. AMS, 2010.
- [Ya3] D. Yafaev, *Quasi-diagonalization of Hankel operators*, to appear in J. d'Analyse Mathématique, arXiv:1403.3941.

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